Class formation in a social network with asset exchange.

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In this paper the study of two kinds of economic exchange, additive and multiplicative, in a system of N agents has been made. The work is divided in two parts, in the first one, the agents are free to interact with each other. The system evolves to a Boltzmann-Gibbs distribution with additive exchange a condenses with a multiplicative one. If bankruptcy is introduced, both types of exchange lead to condensation. Condensation times have been studied. In the second part, the agents are placed in a social network. We analyze the behavior of wealth distributions in time, and the formation of economic classes was observed for certain values of network connectivity.

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I. INTRODUCTION

We have had in the past years a large amount of literature dealing with the study of the distribution of wealth in agent based models with various kinds of interaction rules [1, 2, 3]. Several distributions such as *Boltzmann - Gibbs*, *Gamma*, or *Pareto* can be obtained according to the different conditions of the models [4, 5, 6]. It is well known that real data analysis from several countries yield a *Boltzmann-Gibbs* distribution for that sector of the population with lowest wealth, who are the majority, and a *Pareto* distribution for the minority of the population with the highest wealths. This particular behavior has been reproduced, to some extent, using different kinds of assumptions [7].

In this article we describe a model where this behavior is obtained and the appearance of social classes is observed. These two phenomena are related as we shall see in the following. We analyze the conditions under which the different types of distributions are obtained and also the conditions necessary for the social classes to appear. In the next section of the paper we describe the interacting model for the several cases that we study. We focus on the behavior of entropy, poverty (defined as the minimum state of wealth, where the amount of money is less than minimum allowed exchange) and wealth distributions in sections III and IV.

In section III the dynamics of the model with and without bankruptcy is investigated and in section IV we study the effect of a network in the system. Finally in the last section we give our conclusions.

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II. INTERACTING MODEL

A. Interacting model with no network

The kind of interactions that we have considered in the present work are those in which the total amount of wealth before and after the interaction is conserved, also known as elastic collisions. That is, if (i,j), are the labels of the two agents involved and their wealths are (w_i,w_j) respectively, then we can write

$$w_i(t+\Delta t) = w_i(t) + \Delta w, \tag{1}$$
$$w_j(t+\Delta t) = w_j(t) - \Delta w. \tag{2}$$

Since wealth is preserved in the interaction, we have that $w_i(t+\Delta t) + w_j(t+\Delta t) = w_i(t) + w_j(t)$. It is worth to mention that, according to which amount we choose for Δw, we can have two distinct processes: additive or multiplicative exchange.

In the additive exchange we have $\Delta w = c$, this means that the exchange money is fixed in the whole process, which produces an interaction with time symmetry, that is, if one interaction is produced, for instance i wins and gets Δw from j, in the next time step it is possible to have the inverse one, obtaining the original situation. Of course, since we are dealing with a system with a large number of agents, the probability of that to occur is very low $(1/4N^2)$. In the multiplicative case since Δw is given by $\Delta w = round(\nu \cdot \min(w_i, w_j))$[25], $0 < \nu < 1$, time symmetry is broken and the system never returns to its original set of values.

B. Interacting model within a network

We introduce a network in the system in which every agent i has a_i links to other agents, such that $1 \leq a_i \leq$

a_{max}, where a_{max} is the highest possible number of links allowed in the dynamics. Since this number is not a constant, according to the way the links were introduced (randomly), the average number of links $\langle a \rangle$ in the network is given by

$$\langle a \rangle = \frac{1}{N} \sum_i a_i. \tag{3}$$

With this in mind the dynamics in the network is of course different from case II A, since agent i will only interact with her a_i links through equations (1) and (2). The simulation begins with a population of N agents. Each one of them have the same initial amount of wealth $\langle w \rangle$. In a Monte Carlo step (MCS or time step), two agents (i,j) are chosen randomly and interact through a bet. In the bet one of the agents is chosen randomly to be the winner (of course the remaining agent is the loser), and an amount of wealth Δw is transferred from the loser to the winner. If the wealth of the loser is less than Δw, the transaction is not done. It is important remark that agents are not allowed to have negative wealth.
In all simulations we have used $N = 500$, $\langle w \rangle = 100$ and 10^2 realizations during 4×10^5 MCS. In the additive exchange the value of c is 20, in the multiplicative exchange $\nu = 0.2$. Within a network the average number of links used were 1.468, 1.952, 2.428 and 10.06 (these values correspond to values of $a_{max} = 2, 3, 4$ and 20, respectively).

III. DYNAMICS WITHOUT NETWORK

A. Exchange without bankruptcy

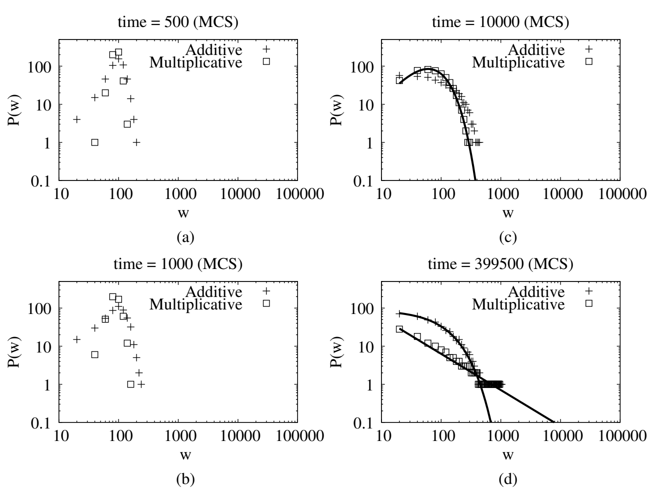

Figure 1: Time evolution of wealth distribution for both cases of exchange (additive and multiplicative).

We studied the wealth distribution evolution with respect to time for both the additive and multiplicative exchange cases. In Figure 1 we display the results, which are not normalized, and find out what it is already reported[1, 8]: in the additive case the system evolves to a stationary state and the distribution is of the *Boltzmann-Gibbs* type. In the multiplicative case the system undergoes different phases including the *Gamma* distribution (Fig 1 (c)) and the *power law* [9, 10, 11] (Fig 1 (d)). Also, as already noted in previous works [12], the system ends in a condensed state, where $N-1$ agents belong to the poverty state, and a single agent is in the maximum state of wealth.
We have also studied the Shannon entropy, defined by

$$S = -\sum_k P_k \ln P_k, \tag{4}$$

where $P_k = n_k/N$, that is, the probability for the agent to be in the k state, with wealth w_k. In Figure 2(a) and 2(b) we display the behavior of the Shannon entropy as well of the poverty for the additive and multiplicative exchanges. In the multiplicative case the entropy reaches a maximum before decreasing monotonically, in consequence, the number of states available to the system is minimum at the condensed state, S_c [26]. In our case, with the parameters given for the system, this value is $S_c = 0.01442$

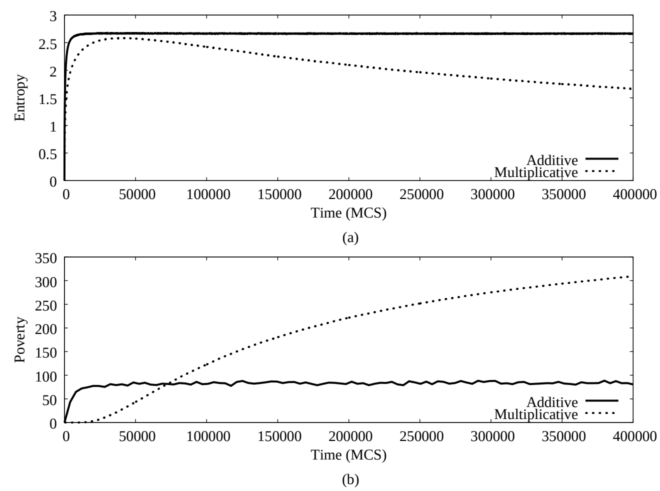

Figure 2: Time evolution of entropy (a) and poverty (b) for both cases of exchange (additive and multiplicative).

B. Exchange with bankruptcy

We say that an agent is bankrupt when it ends up with no money to make more bets; in that case, it is not allowed to participate in the dynamics. As expected, the condensation point is reached faster than in the previous cases when bankruptcy was not included. In those cases, the agents who have reached the absolute poverty were still allowed to participate in the game, and in principle they could become rich, even though this situation was very unlikely.

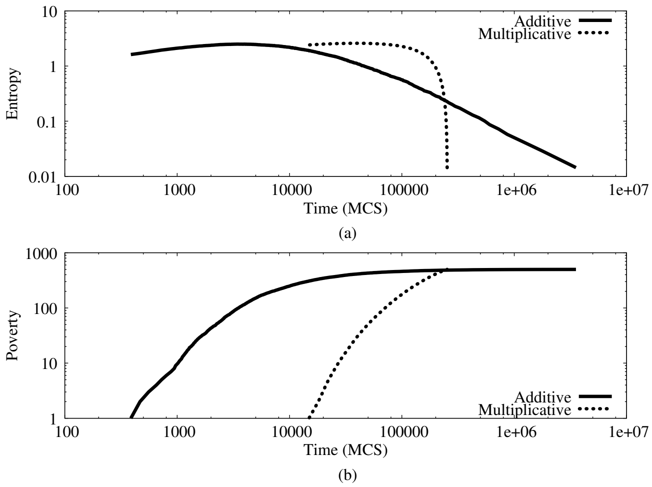

Figure 3: Time evolution of entropy (a) and poverty (b) for both cases of exchange (additive and multiplicative), using the bankruptcy concept.

Figure 3 shows the results with bankruptcy. It is important to note that in both cases, additive and multiplicative, the system reaches the condensed phase. Poverty reaches its maximum value, $N-1$ or 499, and entropy ends up at the S_c value previously given. Since in both cases we obtain condensation, we consider the time it takes to reach this point when we vary parameters such as N, the number of agents, and Δw, the fraction of the wealth exchange. In Figure 4 we show this variation with respect to N and found a different behavior depending of the two cases. In the additive case the dependence of the condensation time with respect to N goes like $t_c^{add} \propto N^3$ whereas in the multiplicative case we get a linear dependence N $t_c^{mul} \propto N$. The dependence of the condensation

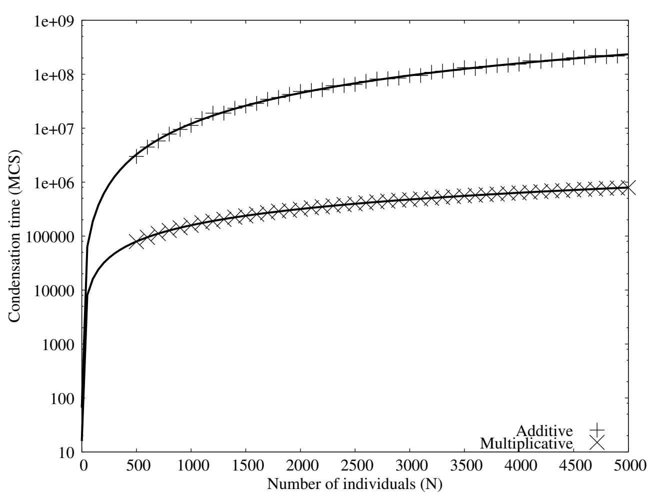

Figure 4: Time of condensation vs number of agents for both cases of exchange (additive and multiplicative), using the bankruptcy concept.

time with Δw is depicted in Figure 5. In the additive case

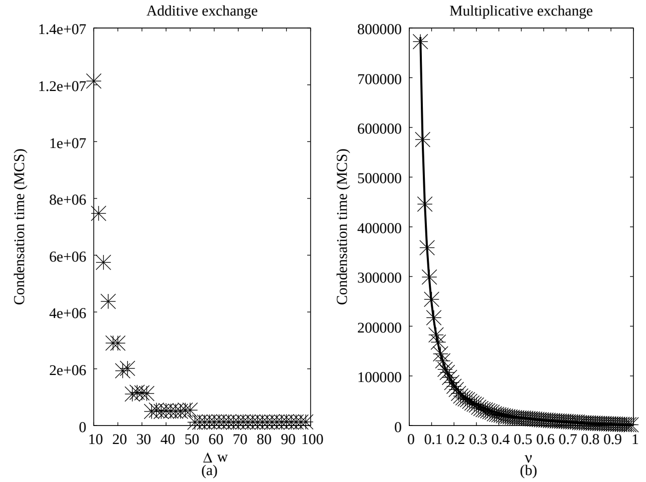

Figure 5: (a) Time of condensation vs Δw (additive), (b) time of condensation vs ν (multiplicative), using the bankruptcy concept.

(fig 5(a)) there is a discrete behavior due to the fact that the condensation time has a well defined value for the different ranges of Δw. This can be explained if we consider the condensation time as being proportional to the minimum of the exchange [27] the two agents can make before one of them reaches absolute poverty $t_c \propto min_{ex}$. For instance if we considered $\langle w \rangle = 100$ and $\Delta w = 50$, two agents (i,j) will have a minimum of exchange $min_{ex} = 2$, that means that there are 2 exchanges at least, before one agent reaches poverty. If we have $\langle w \rangle = 100$ and $\Delta w = 51$, the minimum of exchange is now $min_{ex} = 1$, and only one bet can be made. Therefore the time of condensation depends of the minimum of exchange.

In a multiplicative exchange (Figure 5(b)) the behavior of the condensation time is of the form $t_c^{mul} \propto 1/\nu^2$ and do not shows the discrete situation of the additive case since the bet is not constant. One would expect that if we increase the amount of the wealth exchange in each transaction, the time it takes to condensate gets lower, and that is what the above relation is telling us.

IV. DYNAMICS OF THE EXCHANGE IN A SOCIAL NETWORK

In this section we consider a system composed of N agents interacting with the same rules as before, but this time they cannot interact freely but instead they have relations that form a social network, that we are about to discuss. One would expect that the dynamics shows some differences with respect to the urn case.

We build the network in the following way. A number of links is chosen randomly by every agent in the population, this number ranges from one up to a maximum value of links a_{max}. Now the agents can do bets only with other agents to whom they are linked, generating the social network.

Since the number of links varies among the agents, we will consider an average value of links, $\langle a \rangle$ (see section II), for a given network. We consider again the additive and multiplicative cases, without taking bankruptcy into account, because as we have seen, its main effect is to favor the system to go into the condensed state.

A. Additive exchange

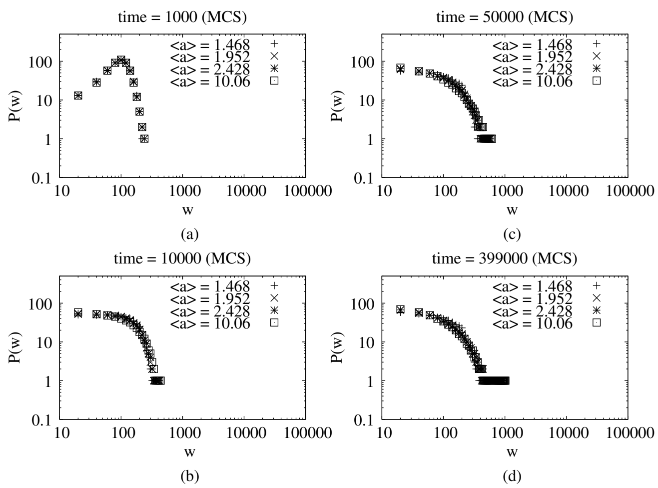

Figure 6: Time evolution of wealth distribution in the additive exchange case for different values of $\langle a \rangle$.

In Figure 6 we display the temporal evolution of the wealth distribution for the additive case and for different values of $\langle a \rangle$. Again the distributions are not normalized since this makes no difference in the discussion. We always get a stationary distribution of the wealth no matter the value of $\langle a \rangle$. The entropy, as well as the poverty, reaches a maximum value (Figure 7). There is also an interesting phenomenon which is the "isle" formation, which consist of groups of linked agents that are isolated from the rest of the network. The exchange becomes isolated from the rest of the system, and the effect of this situation is to produce economic classes as the system evolves. (See inset of Figure 7). This effect is more easily obtained when $\langle a \rangle$ is lower, as one would expect. Therefore, the isolated agents that form the isle are decoupled from the system's dynamics, and they never reach absolute poverty. The more isles we get the less poverty we obtain.

Figure 8 shows wealth distributions for additive exchange for different values of $\langle a \rangle$. For the lower value of $\langle a \rangle = 1.468$, we noticed two social classes, one where the wealth of the agents is below $2\langle w \rangle$ and with a *Boltzmann - Gibbs* distribution (continuous line Figure 8(a)) of the form

$$P(w) = Ce^{-\frac{w}{k\langle w \rangle}}, \qquad (5)$$

where k is equal to 2. The other class, with wealth above

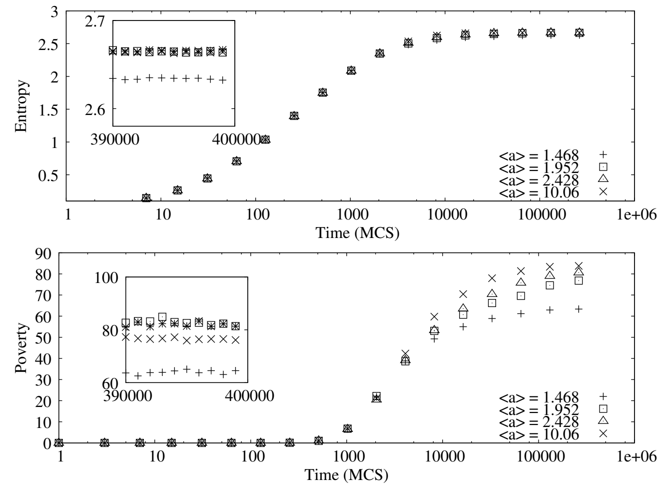

Figure 7: Time evolution of entropy (a) and poverty (b) in the additive exchange case for different values of $\langle a \rangle$.

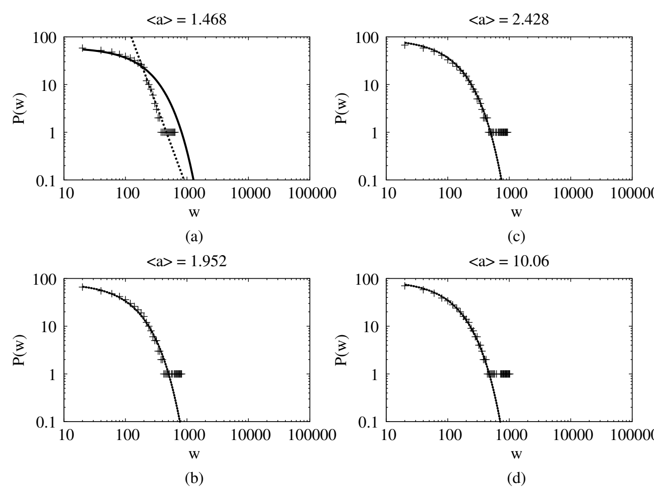

Figure 8: Wealth distribution in the additive exchange case, in time step 399000 MCS for different values of $\langle a \rangle$.

$2\langle w \rangle$ behaves like a Pareto distribution (dashed line Figure 8(a)) of the form

$$P(w) = \frac{C}{x^{1+\alpha}}, \qquad (6)$$

with $\alpha = 2.5$. These two distributions are separated at a value of wealth $w = 2\langle w \rangle$, and it is important to mention that these classes coexist. Other authors have found a transition from one distribution to another as the system evolves [7], but not at the same time. These formation of two classes arises from the fact that at a value of $\langle a \rangle = 1.468$, most of the isles are comprised of two agents. Therefore, the poor sector of the system is formed mostly by isles of 2 agents, while the rich sector have to the possibility to be formed by isles of 3 or more agents, which allows an agent to have a wealth higher than $2\langle w \rangle$.

When the isles formed by two agents start to disappear (figures 8b-8d), also the two classes do vanish. Furthermore when the value of $\langle a \rangle$ changes, also the value of k (eq (5)) does, as shown in table I.

$\langle a \rangle$	k
1.952	1.15
2.428	1.1
10.06	1.05

Table I: Values of k for different average links in the additive exchange case

Note that the systems reaches equilibrium when the "temperature" or $k \langle w \rangle$ is higher than in the non-network case, which has $k = 1$. As $\langle a \rangle$ increases, the value of k tends to 1, which makes sense since a very large value of $\langle a \rangle$ is equivalent to having no network present.
It is important to remark that the formation of the isles in the network allows the exchange transactions to occur in a local way and that prevents a population of agents to become poor. The agents are in a *Boltzmann - Gibbs* distribution, since in the isles the maximum possible wealth is $2 \langle w \rangle$. Therefore isle formation prevent agents from becoming poor, but also rich.

B. Multiplicative exchange

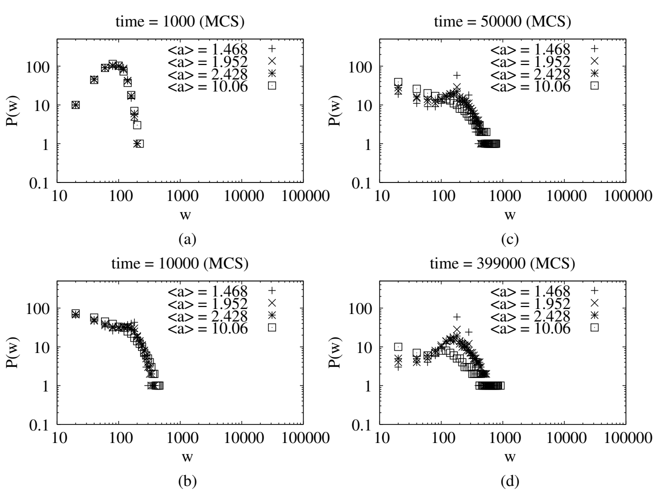

Figure 9: Time evolution of wealth distribution in the multiplicative exchange case for different values of $\langle a \rangle$.

In Figure 9 we show the time evolution of the wealth distribution in a multiplicative exchange process for different values of $\langle a \rangle$. One can observe that in the same way as in the additive case, the distributions are very similar in the first time steps, however, they soon differentiate from each other. For instance, for a time of 10000 MCS, the wealth distributions are different at least for three of the four parameters of $\langle a \rangle$ (1.468, 2.428 and 10.16). It is also important to note that these distributions are not stationary when the entropy is at its maximum value, but they are at later times at values shown in Figure 10. In Figure 10 one can see that entropy goes through a maximum before decreasing and reaching a constant value that is different for each value of $\langle a \rangle$. The system with larger parameter $\langle a \rangle = 10.06$ is the one with more possibilities to condense, as in the non-network case. As one can see from Figure 10(a) and 10(b), its entropy is the lowest and its poverty the highest.

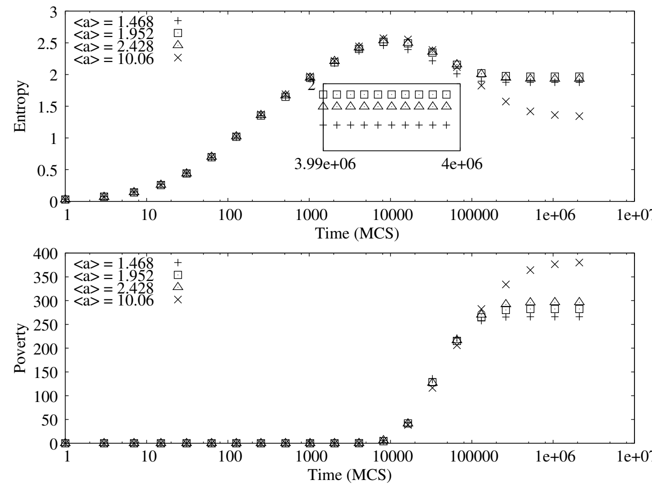

Figure 10: Time evolution of entropy (a) and poverty (b) in the multiplicative exchange case for different values of $\langle a \rangle$.

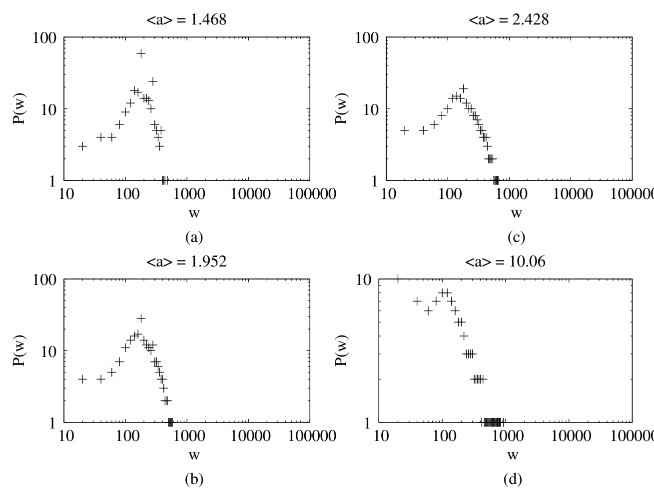

Figure 11: Wealth distribution in the multiplicative exchange case, in time step 399000 MCS for different values of $\langle a \rangle$. Note that $\langle w \rangle = 100$.

Figure 11 shows the wealth distributions in a multiplicative exchange at the same time step ($t = 399000$ MCS) for different values of $\langle a \rangle$. A very interesting phenomenon can be observed in Figure 11(a), the appearance of three well defined classes with wealth of zero (not visible in the

figure), $1.8\langle w\rangle$ and $2.8\langle w\rangle$ respectively (isolated points in the figure). These classes of wealth will eventually become 0, $2\langle w\rangle$ and $3\langle w\rangle$ when the system reaches the stationary state at later times. The isolated point of wealth $1.8\langle w\rangle$ corresponds to the presence of isles of two agents in the system, because, if two agents interact only with each other, the maximum value of wealth that a agent will have is $2\langle w\rangle$. Similarly, the point with wealth equal to $2.8\langle w\rangle$ corresponds to the presence of closed groups of three agents, where an agent will have the possibility to get at most $3\langle w\rangle$ of money. In addition to these classes, we observe a background of agents with variable wealths. These background arises from agents that do not belong to closed groups. In this case there is a larger flux of money among many agents in the network.

It is clear that classes appear due to the formation of isles. A local condensation occurs in each isle, and this gives us a hint about the way the system behaves when the value $\langle a\rangle$ decreases: the entropy of the system should reach a minimum which is stationary and in this point half of the agents are poor and the other half has a wealth of $2\langle w\rangle$. As soon as $\langle a\rangle$ starts to increase, the separation between the classes begins to disappear, until we are left with a class of agents with wealth 0, and another class on non-zero wealth agents, as seen in Figure 11(d). A Similar bimodal behavior has been obtained in Ref [5].

V. SUMMARY AND CONCLUSIONS

In this work we studied an asset exchange model with additive or multiplicative exchanges. We consider the cases when bankruptcy is introduced and also the interaction of agents through a social network. As already known, the additive, non-network case leads to a *Boltzmann - Gibbs* distribution, while in the multiplicative, non-network case, the system evolves to a condensed state where a single agent gets all the money. When we allow for bankruptcy, condensation is obtained for both types of exchange, again in the non-network case. We studied these condensation times, finding that the in the multiplicative exchange the system condenses faster than in the additive case. The time of condensation is proportional to the number of agents. In the additive case the proportionality goes as $t_c^{add} \propto N^3 \cdot min_{ex}$ where, min_{ex} is the minimum exchange two agents can have before one of them reaches poverty. For the multiplicative case the condensation time goes as $t_c^{mul} \propto N/\nu^2$.

From the analysis presented for the wealth distributions we have obtained information about their behavior for different values of time and average value of links, $\langle a\rangle$. We analyzed two different processes, the multiplicative and the additive cases, with and without social network among the interacting agents. When we introduced the social network we obtain stationary distributions in the additive case with a clear formation of coexisting classes, the poor sector of the system following a Boltzmann-Gibbs distribution until a value of wealth $2\langle w\rangle$ and a Pareto distribution from there on.

In the multiplicative case, there is also formation of three well defined classes that arise from the formation of isles in the network, and also from groups of 3 or more agents isolated from the rest of the network. These isles and groups begin to disappear when the parameter $\langle a\rangle$ is increased, since the connectivity of the network is higher. In this later case we expect a global condensation to occur a very high times.

Isle formation in general allows the presence of social classes, since they prevent the money to flow in the entire network. This has as a consequence that in some parts of the system the wealth can not be larger than a maximum amount fixed by the number of members in the isle, effectively preventing global condensation. The number of isles is higher as $\langle a\rangle$ diminishes. In a real society, the flow of money is affected by a myriad processes that cannot be accounted for in a model. All these processes together lead to the formation of classes in a real economy, however, it is important to model some of the phenomena, since this will give us some insights into the complex whole that make up the total behavior of economic activity. The model developed in this work has the virtue of producing an interesting phenomenon like the formation of economic classes, despite the fact that it is so simple and does not take into account the whole economic complexity. We have also observed a correlation between isles and the formation of classes, so we would expect that isle economies would be present in real economies; in other words, sectors of the population whose interaction with the economy is negligible. This point could be explored in a future work.

Acknowledgments

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[25] We used $round(x)$ as the integer closest to x.
[26] The entropy of condensation S_c is calculated as:

$$S_c = -\left(\frac{N-1}{N}\ln\frac{N-1}{N} + \frac{1}{N}\ln\frac{1}{N}\right)$$

[27] The minimum of exchange is given by

$$min_{ex} = \left\lfloor \frac{\langle w\rangle}{\Delta w} \right\rfloor$$